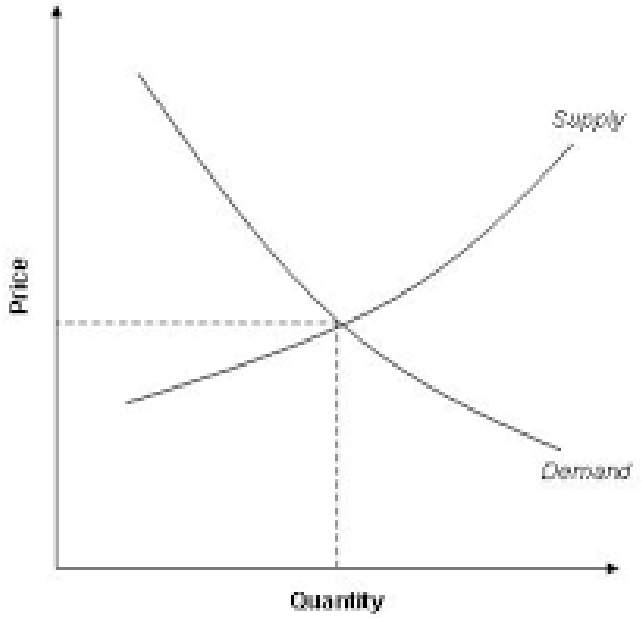
QUANTITATIVE FINANCE

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Preference/Choice

- For any product which has a price and quantity there is an equilibrium point based on which this price and quantity is found



Preference/Choice

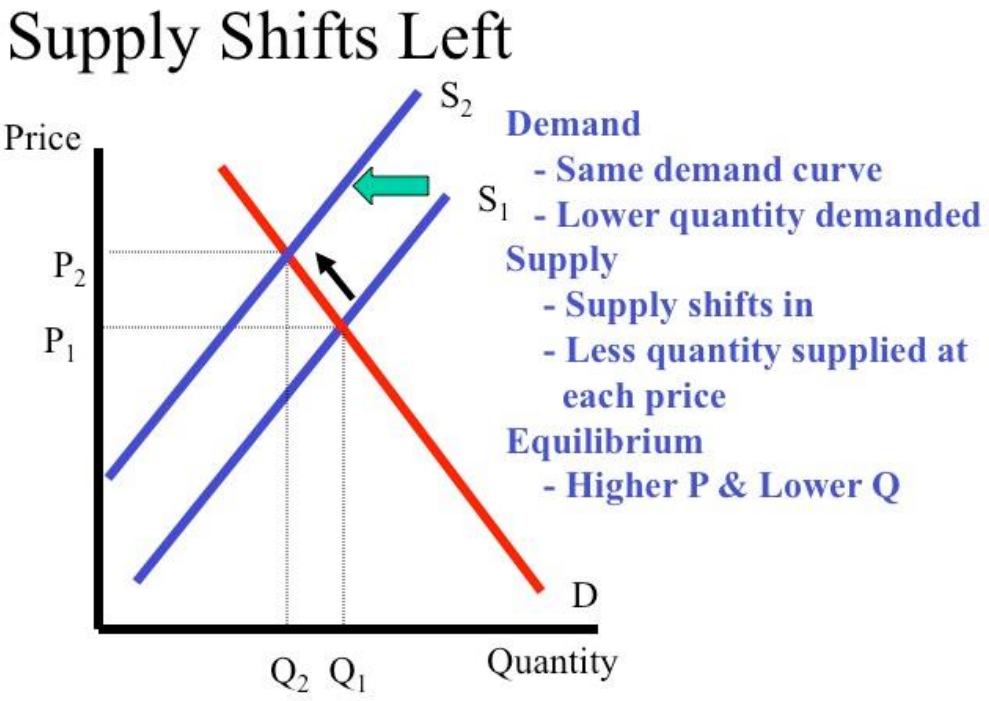
- Marginal revenue/return from a decision/financial investment is the change in revenue/return associated with one unit change in out put (O/P).
- Marginal revenue is zero at the quantity that generates maximum total revenue, and is negative beyond that point.
- For linear demand curve, he absolute value of he slope of the marginal revenue cruve is twice that of the demand curve.

Preference/Choice

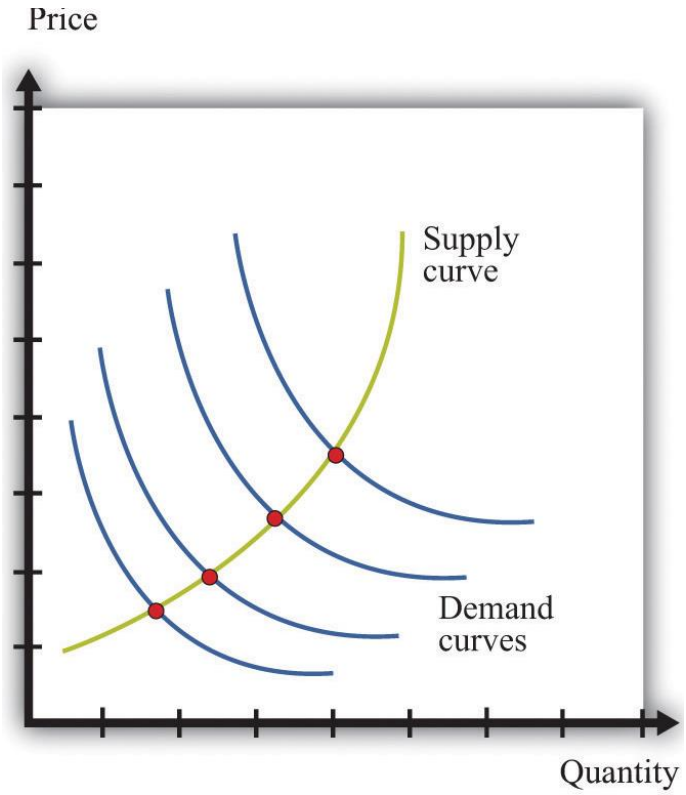
- In trying to find the price and quantity for the demand curve one needs to know price elasticity of demand, which is the percentage change in quantity demanded divided by the percentage change in price.
- Price elasticties can be found only using two approaches which are point and arc elasticity.
- It is useful to classify demand relationships on the basis of price elasticity
- Thus

If	**Demand**
❖ Price elasticity of demand < -1	Elastic
❖ Price elasticity of demand = -1	Unitary Elastic
❖ -1 < Price elasticity of demand ≤ 0	Inelastic

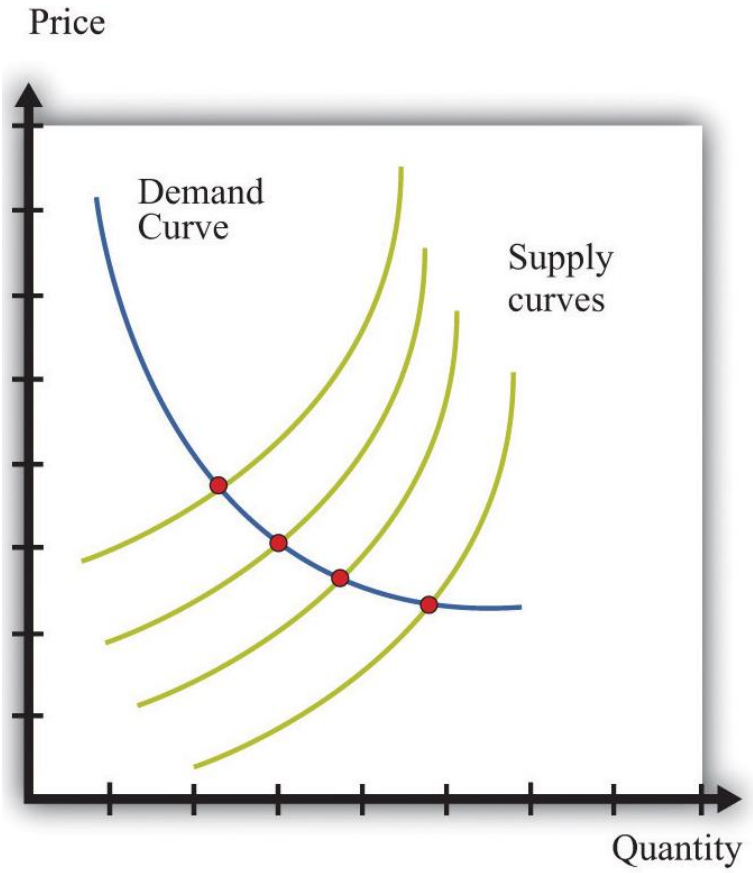
Preference/Choice



Preference/Choice



Preference/Choice

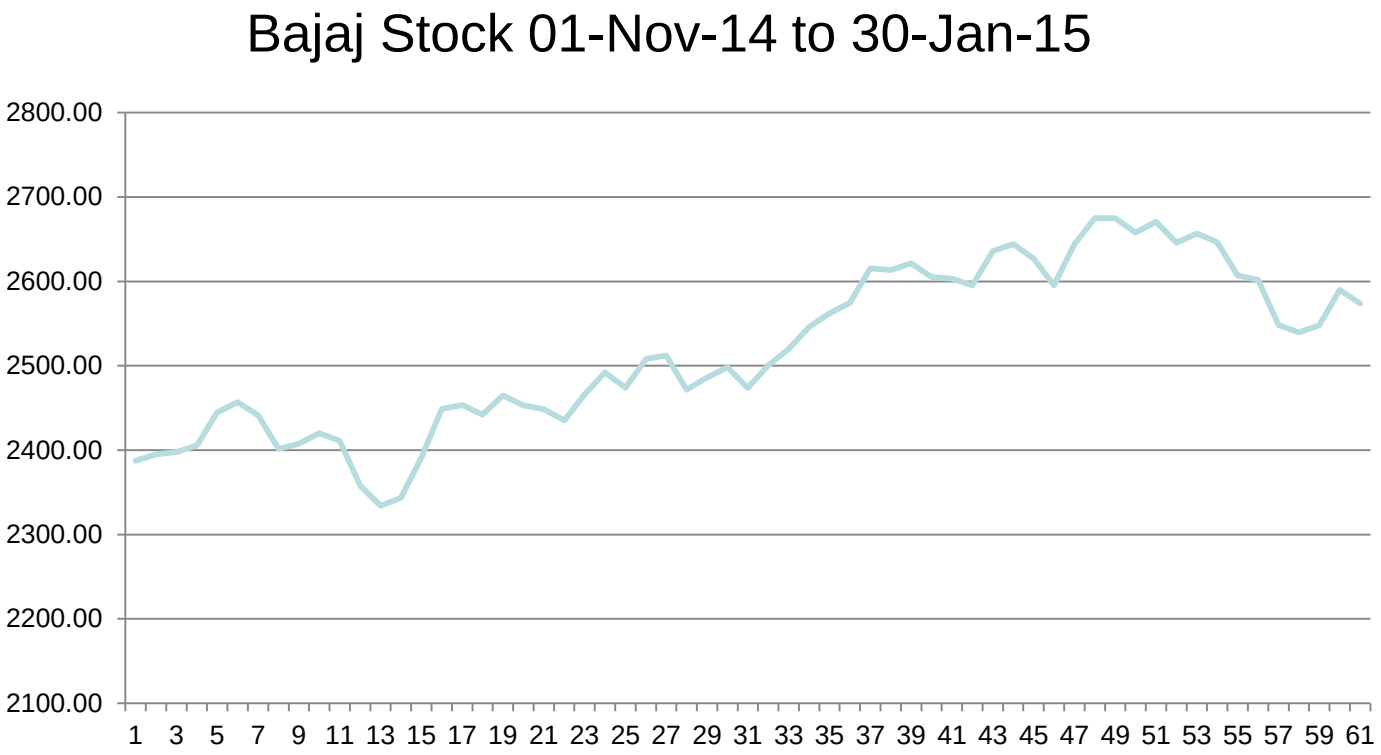


Investment Environment

We will later on learn about

- Securities (financial securities, e.g., stock, options, etc).
- Treasury bills, T-Bill of 90 day sin India, in this context it would be good to introduce the term related to risk free interest rate, r_f.
- Long term bonds and their duration.
- Common stocks

Investment Environment



Investment Environment

- Risk, Return, Diversification
- Security markets and how hey operate
- Financial Intermediaries

Utility

Utility analysis

Different investors have different attributes and. Some may have the property of risk loving, others are risk neutral while a third category is risk averse. Each investor has with him/her an opportunity set. This opportunity set is specific to that person only.

Utility

A		B	
Outcome value(i)	P[i]	Outcome value(i)	P[i]
15	1/3	20	1/3
10	1/3	12	1/3
15	1/3	8	1/3

In reality what would a person do if he or she has two outcome sets in front of him/her.

For A we have the expected value of outcome as 13.33 and for B also it is 13.33

Utility

A		B	
Outcome value(i)	P[i]	Outcome value(i)	P[i]
15	½	20	1/3
10	¼	12	1/3
15	¼	8	1/3

Now for A we have the expected value of outcome as 13.75 and for B it is still 13.33.

Utility

Outcome	Team X	Team Y
Wins	40	45
Draws	20	5
Losses	10	20

Case I		Case II	
Outcome	Points	Outcome	Points
Win	2	Win	5
Draw	1	Draw	1
Lose	0	Lose	0

Utility

Case I

Team A = 100; Team B = 95, which means A > B, i.e., A is ranked higher than B.

Case II

Team A = 220; Team B = 230, which means B > A, i.e., B is ranked higher than A.

Utility

On a general nomenclature we should have the expected value or utility given by

$$E[U] = \sum_{\forall W} U(W) \frac{N(W)}{\sum_{\forall W} N(W)}$$

here U(W) is the utility function which is a function of the wealth, W, while N(W) is the number of outcomes with respect to a certain level of income W.

Utility

Remember in general utility values cannot be negative, but many function may give negative values.

For analysis to make the problem simple we may consider the value to be zero even though in actuality it is negative.

Utility

Consider an example where a single individual is facing the same set of outcomes at any instant of time but we try to analyze his/her expected value addition or utility separately based on two different utility functions

1) U[W(1)] = W(1) +1

2) U[W(2)] = $W(2)^2$ + W(2)

Outcome	W(1)	U[W(1)]	P(W(1)	W(2)	U[W(2)]	P(W(2)
15	1.5	2.5	0.15	1.5	3.75	0.15
20	2.0	3.0	0.20	2.0	6.00	0.20
25	2.5	3.5	0.25	2.5	8.75	0.25
10	3.0	4.0	0.10	3.0	12.00	0.10
5	0.5	1.5	0.05	0.5	0.75	0.05
25	5.0	6.0	0.25	5.0	30.00	0.25

Accordingly we have E[U(1)] = 3.825 and E[U(2)] = 12.69. So we can have a different decision depending on the form of utility function we are using.

Investment Process

Now we have two different utility functions used one at a time for two different decisions

1) U[W(1)] = W(1) - 5 and

2) U[W(2)] = 2*W(2)-W(2)$^{1.25}$

Outcome	W	U[W(1)]	U[W(2)]	Decision (A)	Decision (B)
8	4	0	2.34	Yes	No
3	5	0	2.52	No	Yes
4	6	1	2.60	No	Yes
6	7	2	2.61	Yes	No
9	8	3	2.54	Yes	No
5	9	4	2.41	No	Yes

For utility function U[W(1)]

U(A,1)=0*8/(8+6+9)+2*6/(8+6+9)+3*9/(8+6+9)=1.69

U(B,1)=0*3/(3+4+5)+1*4/(3+4+5)+4*5/(3+4+5)=2.00

For utility function U[W(2)]

U(A,2)=2.34*8/(8+6+9)+2.61*6/(8+6+9)+2.54*9/(8+6+9) ≅2.50

U(B,2)=2.52*3/(3+4+5)+2.60*4/(3+4+5)+2.41*5/(3+4+5) ≅2.50

Investment Process

A venture capitalist is considering two possibilities of investment. The first alternative is buying government treasury bills which cost Rs. 6,00,000. While the second alternative has three possible outcomes, the cost of which are Rs.10,00,000, Rs. 5,00,000 and Rs. 1,00,000 respectively. The corresponding probabilities are 0.2, 0.4 and 0.4 respectively. If we consider the power utility function $U(W)=W^{1/2}$, then the first alternative has a utility value of Rs.776 while the second has an expected utility value of Rs. 609. Hence the first alternative is preferred.

Utility

Would the above problem give a different answer if we used an utility function of the form $U(W) = W^{1/2} + c$ (where c is a positive o a negative constant)?

Utility

In a span of 6 days the price of a security fluctuates and a person makes his/her transactions only at the following prices. We assume U[P] = ln(P)

Day	P	U[P]	Number of Outcomes	Probability
1	1000	6.91	35	0.35
2	975	6.88	20	0.20
3	950	6.86	10	0.10
4	1050	6.96	15	0.15
5	925	6.83	5	0.05
6	1025	6.93	15	0.15

Expected utility is 6.91

If $U[P]= P^{0.25}$, then expected utility is 33.63

Utility

General properties of utility functions

1) Non-satiation: The first restriction placed on utility function is that it is consistent with **more being preferred to less**. This means that between two certain investments we always take the one with the largest outcome, i.e., $U(W+1) > U(W)$ for all values of W. Thus $dU(W)/dW > 0$

Utility

2) If we consider the investors or the decision makers perception of absolute risk, then we have the concept/property of (i) risk aversion, (ii) risk neutrality and (iii) risk seeking. Let us consider an example now

Utility

Invest	Prob	Do not invest	Prob
2	½	1	1
0	½		

Price for investing is 1 and it is a fair gamble, in the sense its value is exactly equal to the decision of not investing

Utility

Thus

- $U(I_1)*P(I_1) + U(I_2)*P(I_2) < U(DI)*1$
  $\rightarrow$ risk averse
- $U(I_1)*P(I_1) + U(I_2)*P(I_2) = U(DI)*1$
  $\rightarrow$ risk neutral
- $U(I_1)*P(I_1) + U(I_2)*P(I_2) > U(DI)*1$
  $\rightarrow$ risk seeker

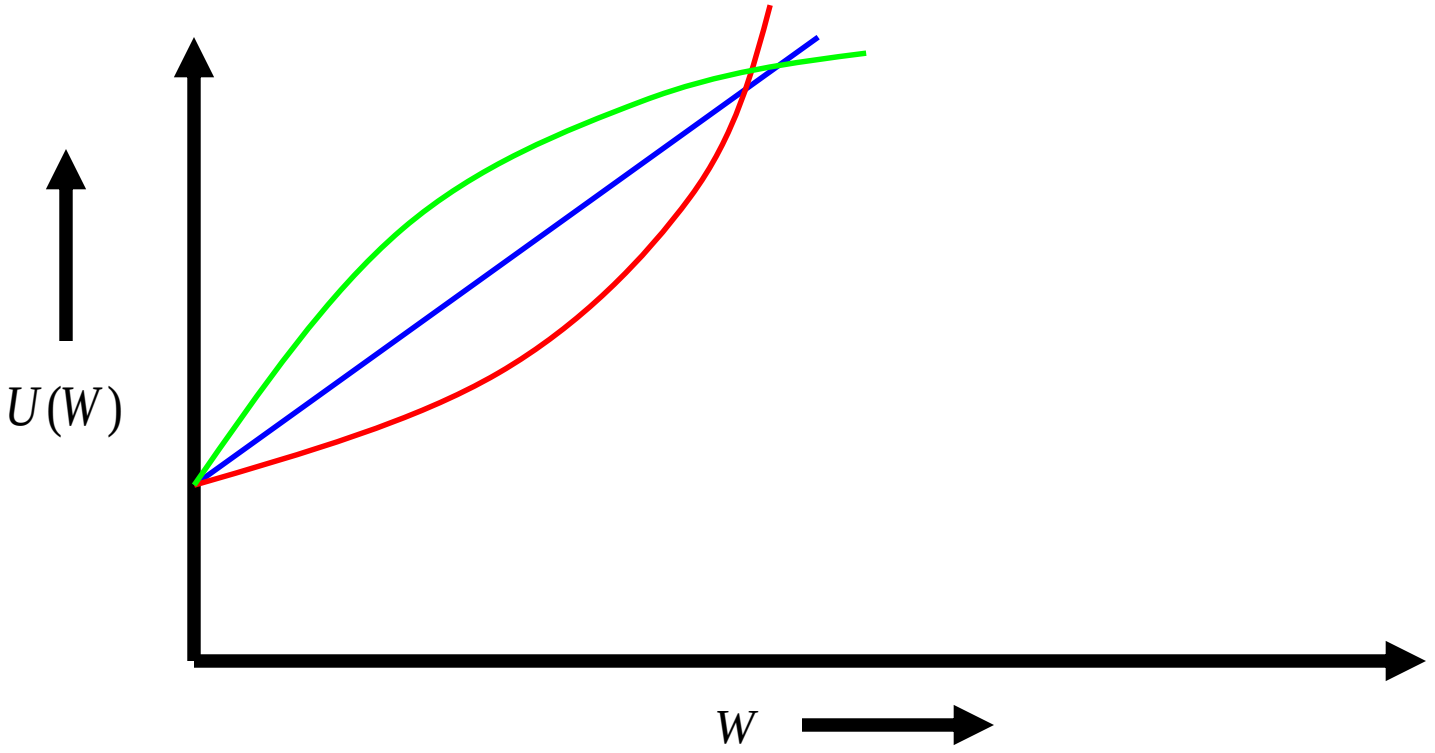
Utility

Another characteristic by which to classify a risk averse, risk neutral and risk seeker person is

- $d^2U(W)/dW^2 = U''(W) < 0 \rightarrow$ risk averse
- $d^2U(W)/dW^2 = U''(W) = 0 \rightarrow$ risk neutral
- $d^2U(W)/dW^2 = U''(W) > 0 \rightarrow$ risk seeker

Utility

Utility curves



Utility

Marginal Utility Function

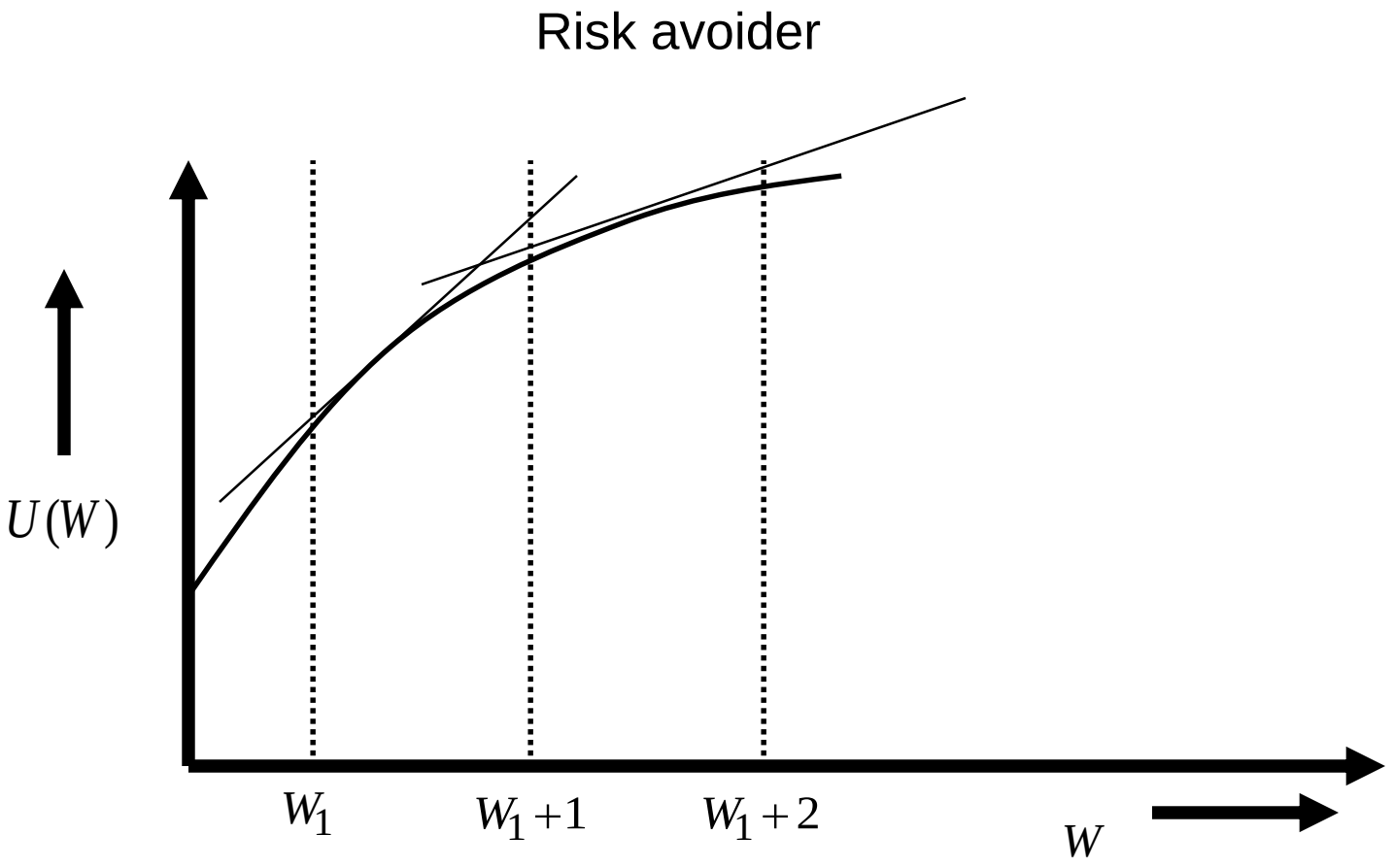
- Marginal utility function looks like a concave function $\rightarrow$ risk averse
- Marginal utility function looks neither like a concave nor like a convex function $\rightarrow$ risk neutral
- Marginal utility function looks like a convex function $\rightarrow$ risk seeker

Utility

Marginal Utility Rate

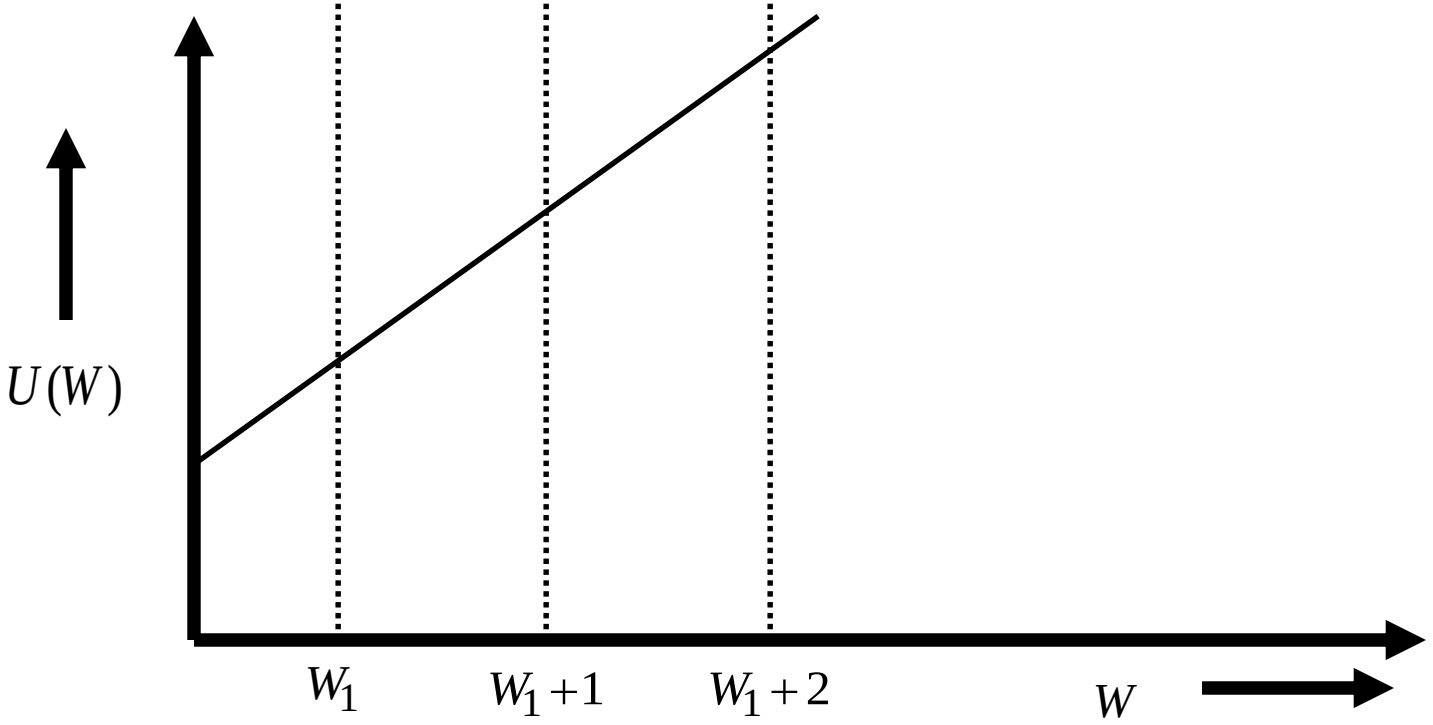
- Marginal utility rate is increasing at a decreasing rate $\rightarrow$ risk averse
- Marginal utility rate is increasing at a constant rate $\rightarrow$ risk neutral
- Marginal utility rate is increasing at a increasing rate $\rightarrow$ risk seeker

Utility

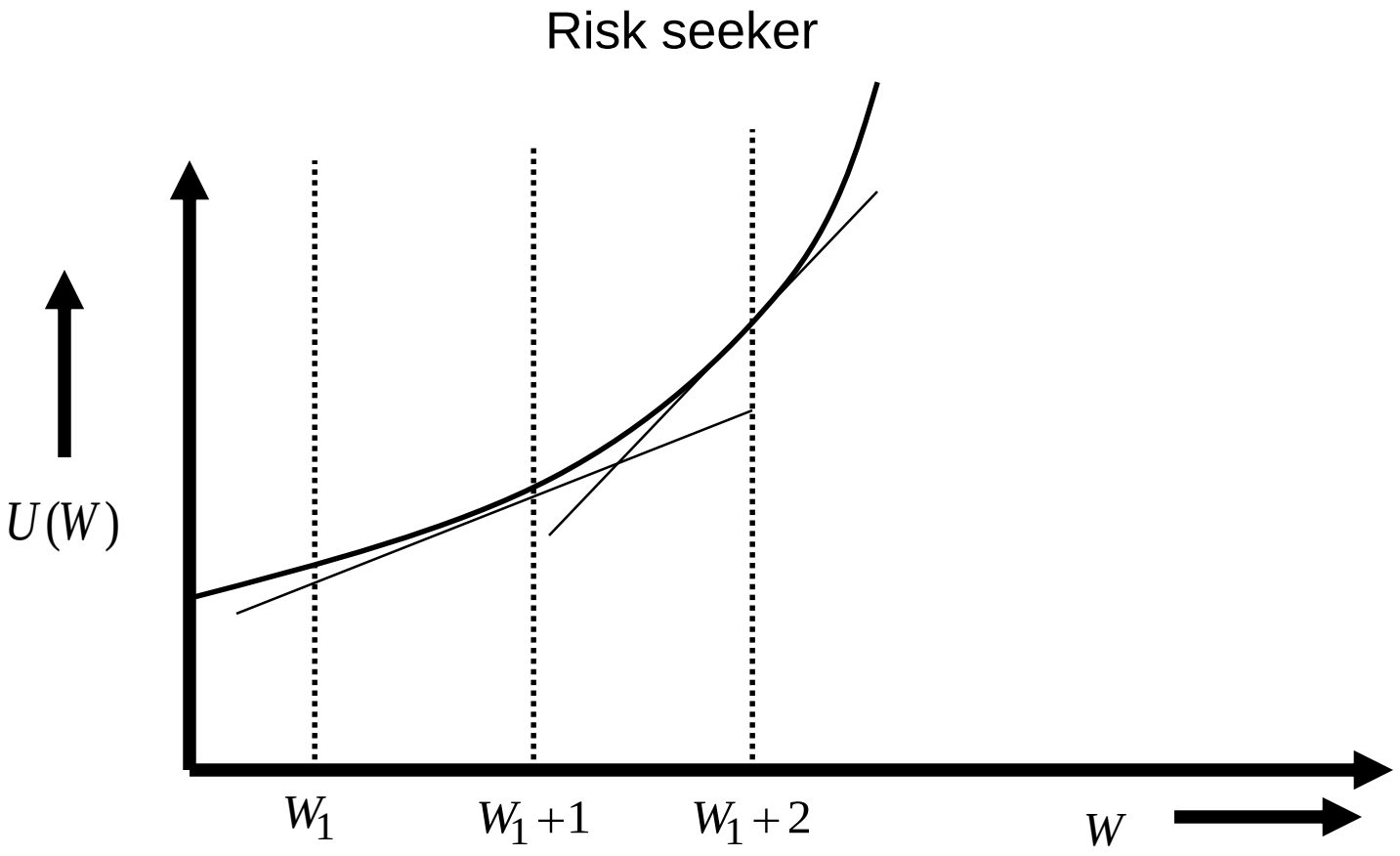


Utility

Risk neutral



Utility



Utility

Few other important concepts

Condition	Definition	Implication
Risk aversion	Reject a fair gamble	$U''(W) < 0$
Risk neutrality	Indifference to a fair gamble	$U''(W) = 0$
Risk seeking	Select a fair gamble	$U''(W) > 0$

Utility

3) Absolute risk aversion property of utility function where by absolute risk aversion we mean

$$A(W) = -[d^2U(W)/dW^2]/[dU(W)/dW]$$

$$= -U''(W)/U'(W)$$

Utility

For the three different types of persons

- Decreasing absolute risk aversion

  $\rightarrow A'(W) = dA(W)/d(W) < 0$
- Constant absolute risk aversion

  $\rightarrow A'(W) = dA(W)/d(W) = 0$
- Increasing absolute risk aversion

  $\rightarrow A'(W) = dA(W)/d(W) > 0$

Utility

	Condition	Definition	Property
1)	Decreasing absolute risk aversion	As wealth increases the amount held in risk assets increases	$A'(W) < 0$
2)	Constant absolute risk aversion	As wealth increases the amount held in risk assets remains the same	$A'(W) = 0$
3)	Increasing absolute risk aversion	As wealth increases the amount held in risk assets decreases	$A'(W) > 0$

Utility

4) Relative risk aversion property of utility function where by relative risk aversion we mean

$$R(W) = - W * [d^2U(W)/dW^2]/[dU(W)/dW]$$

$$= - W * U''(W)/U'(W)$$

Utility

For the three different types of persons

- Decreasing relative risk aversion

  $\rightarrow R'(W) = dR(W)/dW < 0$
- Constant relative risk aversion

  $\rightarrow R'(W) = dR(W)/dW = 0$
- Increasing relative risk aversion

  $\rightarrow R'(W) = dR(W)/dW > 0$

Utility

	Condition	Definition	Property
1)	Decreasing relative risk aversion	As wealth increases the % held in risky assets increases	$R'(W) < 0$
2)	Constant relative risk aversion	As wealth increases the % held in risky assets remains the same	$R'(W) = 0$
3)	Increasing relative risk aversion	As wealth increases the % held in risky assets decreases	$R'(W) > 0$

Utility

Some useful utility functions

- Quadratic: $U(W) = W - b*W^2$ (b is a positive constant)
- Logarithmic: $U(W) = \ln(W)$
- Exponential: $U(W) = - e^{-aW}$ (a is a positive constant)
- Power: $c*W^c$ ($c \leq 1$ and $c \neq 0$)

Utility

$$U(W) = W - b*W^2$$

Then:

- $A'(W)=4*b^2/(1- 2*b*W)^2$
- $R'(W)=2*b/(1- 2*b*W)^2$

Hence we use this utility function for people with

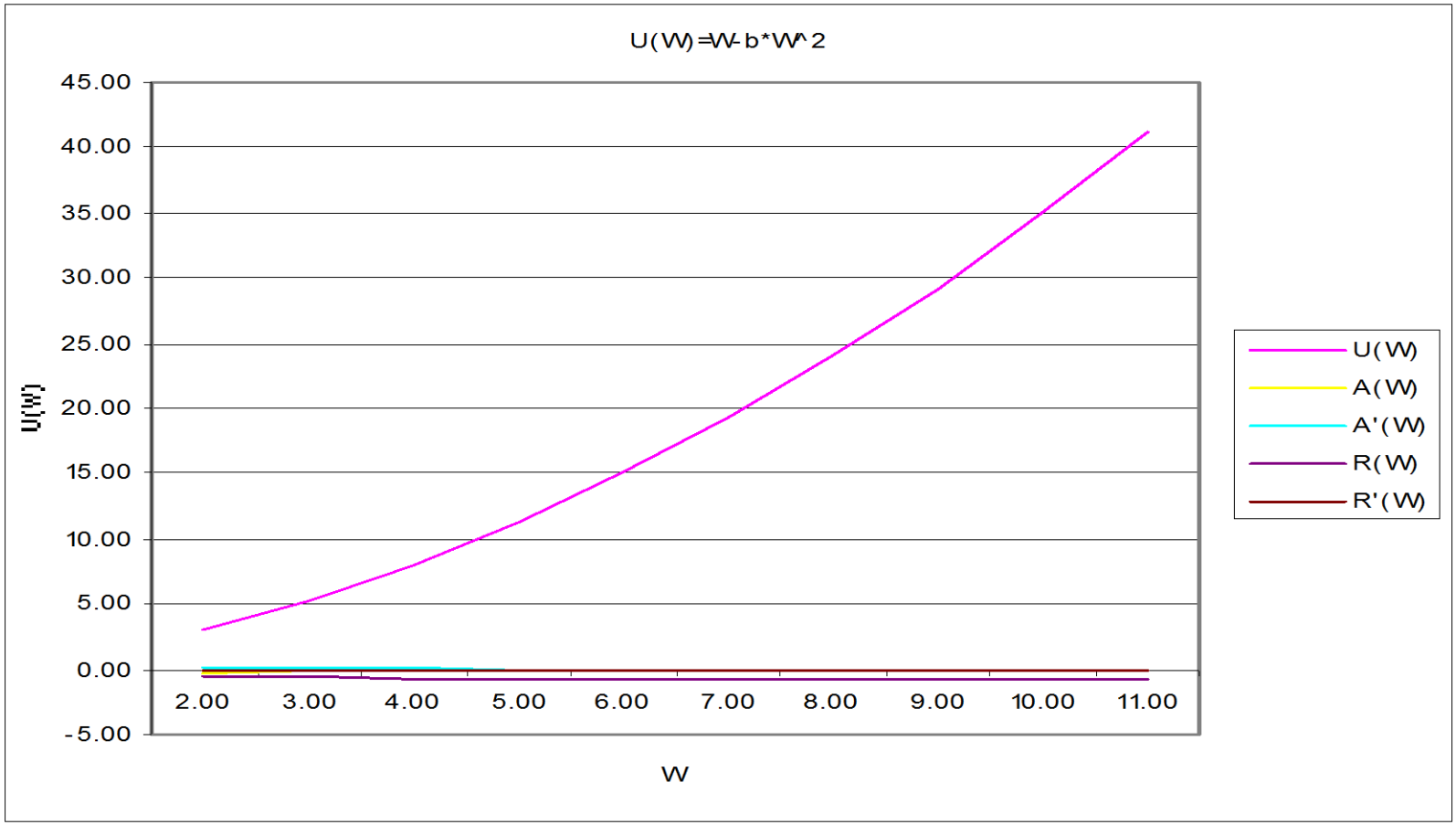
(i) increasing absolute risk aversion and

(ii) increasing relative risk aversion.

Utility

W	W-b*W^2	A(W)	A'(W)	R(W)	R'(W)
2.00	3.00	-0.25	0.06	-0.50	-0.13
3.00	5.25	-0.20	0.04	-0.60	-0.08
4.00	8.00	-0.17	0.03	-0.67	-0.06
5.00	11.25	-0.14	0.02	-0.71	-0.04
6.00	15.00	-0.13	0.02	-0.75	-0.03
7.00	19.25	-0.11	0.01	-0.78	-0.02
8.00	24.00	-0.10	0.01	-0.80	-0.02
9.00	29.25	-0.09	0.01	-0.82	-0.02
10.00	35.00	-0.08	0.01	-0.83	-0.01
11.00	41.25	-0.08	0.01	-0.85	-0.01

Utility


U(W)=W-b*W^2
45.00
40.00
35.00
30.00
25.00
20.00
15.00
10.00
5.00
0.00
-5.00
U(W)
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00
11.00
W
U(W)
A(W)
A'(W)
R(W)
R'(W)

Utility

$$U(W) = \ln(W)$$

Then:

- $A'(W) = -1/W^2$
- $R'(W) = 0$

We use this utility function for people with

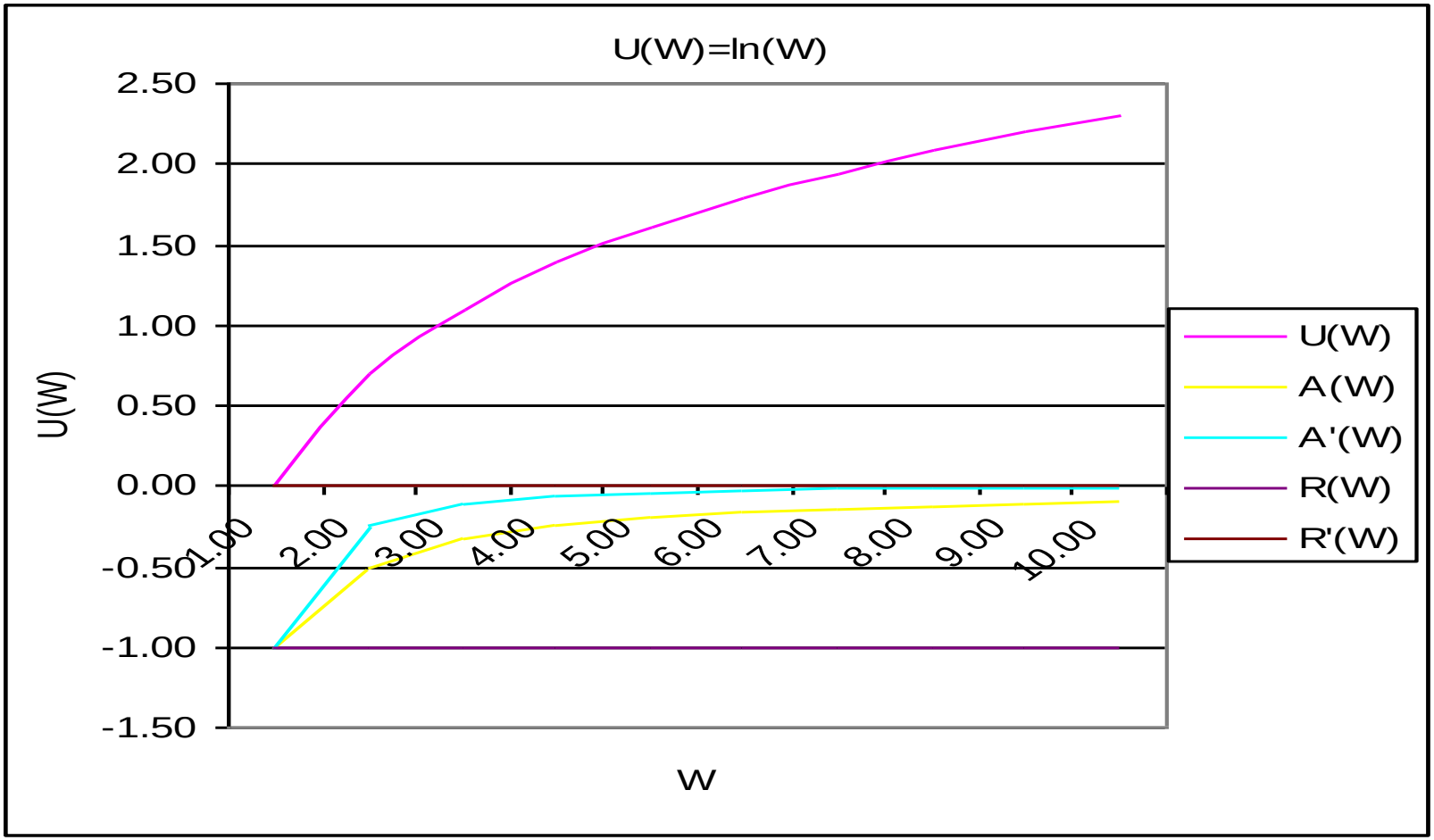
(i) decreasing absolute risk aversion and

(ii) constant relative risk aversion

Utility

W	ln(W)	A(W)	A'(W)	R(W)	R'(W)
1.00	0.00	-1.00	-1.00	-1.00	0.00
2.00	0.69	-0.50	-0.25	-1.00	0.00
3.00	1.10	-0.33	-0.11	-1.00	0.00
4.00	1.39	-0.25	-0.06	-1.00	0.00
5.00	1.61	-0.20	-0.04	-1.00	0.00
6.00	1.79	-0.17	-0.03	-1.00	0.00
7.00	1.95	-0.14	-0.02	-1.00	0.00
8.00	2.08	-0.13	-0.02	-1.00	0.00
9.00	2.20	-0.11	-0.01	-1.00	0.00
10.00	2.30	-0.10	-0.01	-1.00	0.00

Investment Process



Utility

$$U(W) = - e^{-aW}$$

Then:

- $A'(W) = 0$
- $R'(W) = a$

We use this utility function for people with

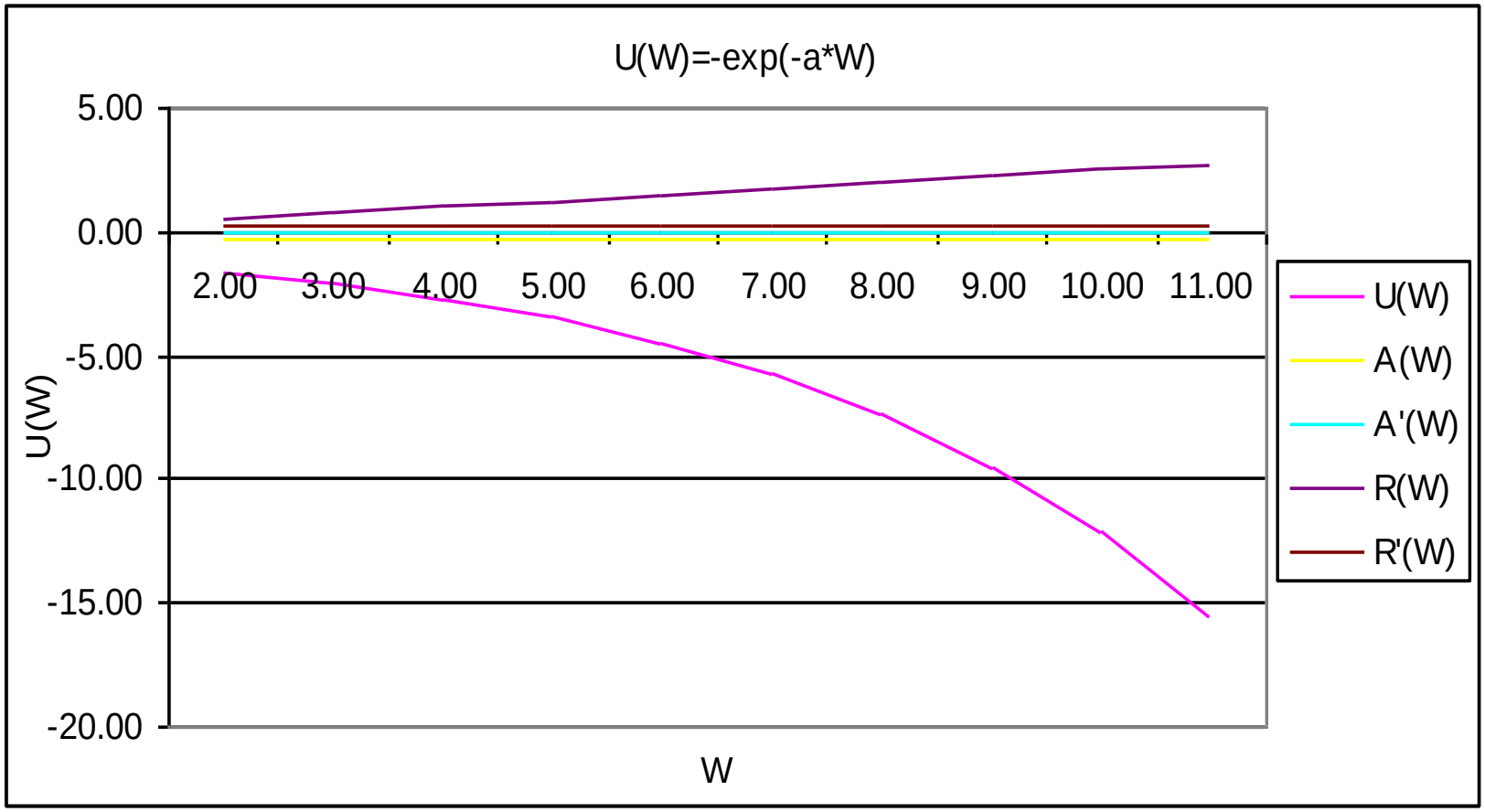
(i) constant absolute risk aversion and

(ii) increasing relative risk aversion.

Utility

W	U(W)	A(W)	A'(W)	R(W)	R'(W)
2.00	-1.65	-0.25	0.00	0.50	0.25
3.00	-2.12	-0.25	0.00	0.75	0.25
4.00	-2.72	-0.25	0.00	1.00	0.25
5.00	-3.49	-0.25	0.00	1.25	0.25
6.00	-4.48	-0.25	0.00	1.50	0.25
7.00	-5.75	-0.25	0.00	1.75	0.25
8.00	-7.39	-0.25	0.00	2.00	0.25
9.00	-9.49	-0.25	0.00	2.25	0.25
10.00	-12.18	-0.25	0.00	2.50	0.25
11.00	-15.64	-0.25	0.00	2.75	0.25

Utility


U(W)=-exp(-a*W)
5.00
0.00
-5.00
-10.00
-15.00
-20.00
U(W)
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00
11.00
W
U(W)
A(W)
A'(W)
R(W)
R'(W)

Utility

$$U(W) = c*W^c$$

Then:

- $A'(W) = (c-1)/W^2$
- $R'(W) = 0$.

We use this utility function for people with

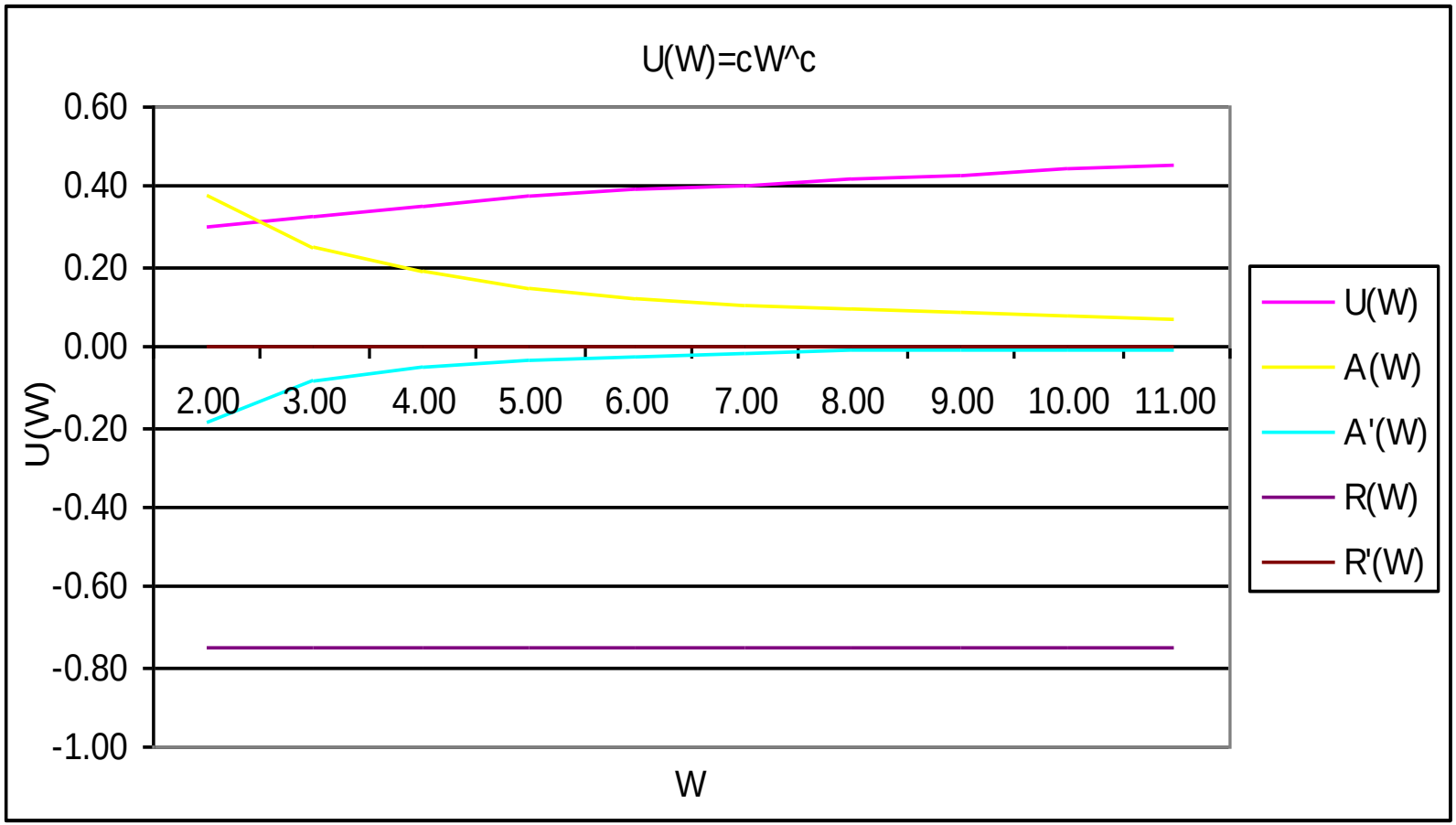
(i) decreasing absolute risk aversion

(ii) constant relative risk aversion.

Utility

W	U(W)	A(W)	A'(W)	R(W)	R'(W)
2.00	0.30	0.38	-0.19	-0.75	0.00
3.00	0.33	0.25	-0.08	-0.75	0.00
4.00	0.35	0.19	-0.05	-0.75	0.00
5.00	0.37	0.15	-0.03	-0.75	0.00
6.00	0.39	0.13	-0.02	-0.75	0.00
7.00	0.41	0.11	-0.02	-0.75	0.00
8.00	0.42	0.09	-0.01	-0.75	0.00
9.00	0.43	0.08	-0.01	-0.75	0.00
10.00	0.44	0.08	-0.01	-0.75	0.00
11.00	0.46	0.07	-0.01	-0.75	0.00

Utility


U(W)=cW^c
0.60
0.40
0.20
0.00
-0.20
-0.40
-0.60
-0.80
-1.00
U(W)
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00
11.00
W
U(W)
A(W)
A'(W)
R(W)
R'(W)

Utility

The actual value of expected utility is of no use, except when comparing with other alternatives. Hence we use an important concept of **certainty equivalent**, which is the amount of certain wealth (risk free) that has the utility level exactly equal to this expected utility value.

We define U(C) = E[U(W)], where C is the certainty value

Utility

How is this value of C useful

- Suppose that we have a decision process with a set of outcomes, their probabilities and the corresponding utility values. In case we want to compare this decision process we can find the certainty equivalent so that comparison is easier.
- To find the exact form of the utility function for a person who is not clear about the form of utility function he/she uses.

Utility

Suppose you face two options. Under option # 1 you toss a coin and if head comes you win Rs. 10, while if tail appears you win Rs. 0. Under option # 2 you get an amount of Rs. M. Also assume that your utility function is of the form $U(W) = W – 0.04*W^2$. It means that after you win any amount the utility you get from the amount you won.

For the first option the expected utility value would be Rs. 3, while the second option has an expected utility of Rs. $M – 0.04*M^2$. To find the certainty equivalent we should have $U(M) = M – 0.04*M^2 = 3$. Thus M = 3.49, i.e., C = 3.49, as U(3.49) = E[U(W)]

Utility

The above example illustrates that you would be indifferent between option # 1 and option # 2.

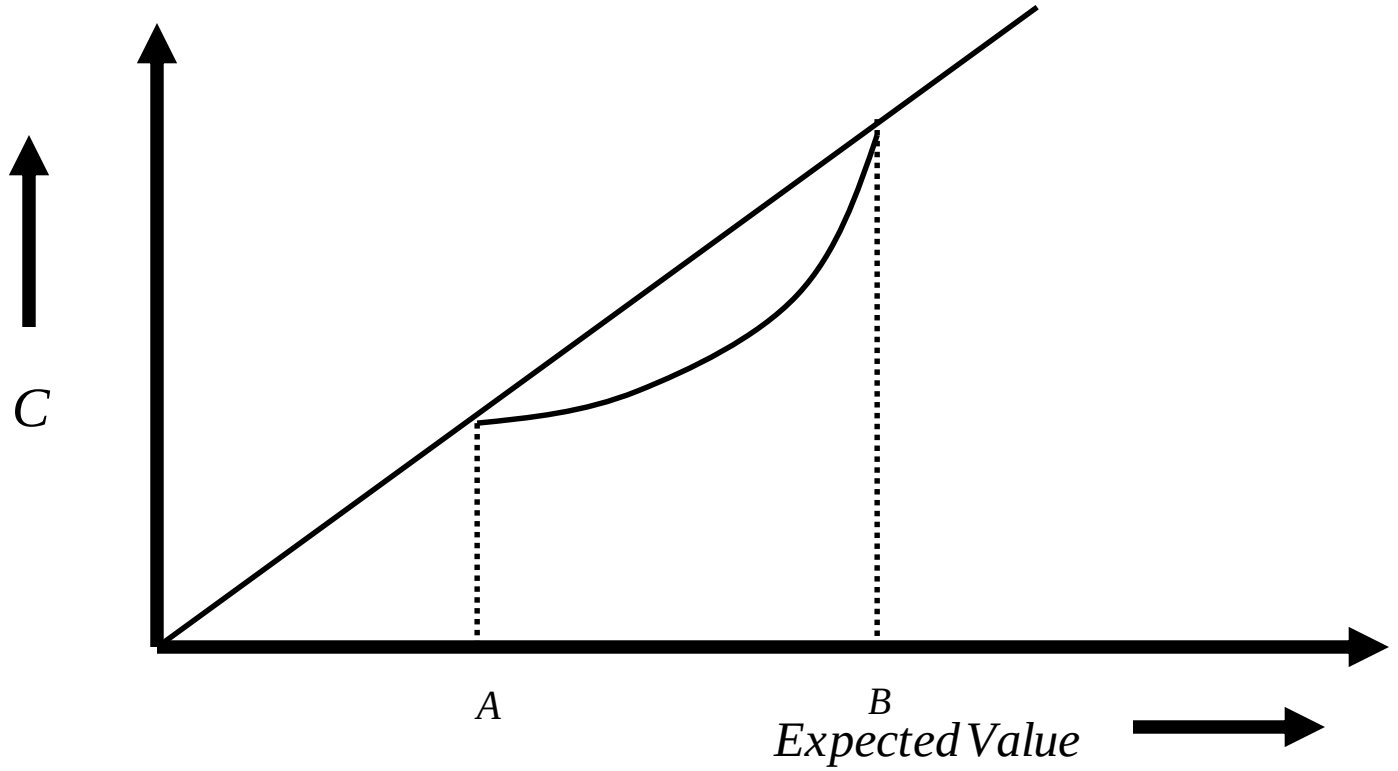
Now suppose if you face a different situation where you have option # 1 as before but a different option # 2 where you get Rs. 5. Then obviously you would choose option # 2 here, as $U(5) = 5 - 0.04*5^2 = 4 > 3.49$.

For the venture capital problem the certainty value for the option # 2 is Rs. 370881, as $U(370881) = 370881^{0.5} = 609$

Utility

- A risk averse person will select a equivalent certain event rather than the gamble
- A risk neutral person will be indifferent between the equivalent certain event and the gamble
- A risk seeking person will select the gamble rather than the equivalent certain event

Utility



Utility

A and B are wealth values, i.e., values of W. Also for ease of our analysis we consider that U(W)=W. Form a lottery such that it has an outcome of A with probability p and the other outcome is B with a probability (1-p). Change the values of p and ask the investor how much certain wealth (C) he/she will have in place of the lottery. Thus C varies with p. Now the expected value of lottery is p*A+(1-p)*B. A risk averse person will have C<p*A+(1-p)*B.

Plot the values of C and you already have the expected values of the lottery.

Utility

How would you find the explicit form of the utility function of a person. Suppose you know that it is of the form $U(W) = - e^{-aW}$. You ask the person that given a lottery which has a 50-50 chance of winning Rs. 1,000,000 or Rs. 4,00,000. In order to buy this lottery what was he/she willing to pay. If the answer is Rs. 4,00,000, it means that the person is indifferent between a certain equivalent amount of Rs. 4,00,000 and the lottery (which is a fair gamble).

Hence $- e^{-400000*a} = 0.5*(-e^{-1000000*a}) + 0.5*(-e^{-100000*a})$. Solving through iteration process we have $a=1.604*10^{-6}$

Investment Process

Axioms of utility functions

1) An investor can always say whether A = B, A > B or A < B
2) If A > B and B > C, then A > C
3) Consider X = Y. Then assume we combine with X with another decision Z, such that X is with P(X) = p and Z is with P(Z) =1-p. On the same lines we have the same decision Z with Y, such that Y is with P(Y) = p and Z is with P(Z) = 1-p. The X+Z = Y+Z
4) For every gamble there is a **certainty equivalent** such that a person is **indifferent** between the gamble and the certainty equivalent

Utility

Comparison between mean-variance and utility function
The utility function used is ($U(W)=W-bW^2$), which is quadratic
Consider we have three assets and the prices are as follows

No	A	B	C	R(A)	R(B)	R(C)	P(i)
1	100	105	80	---	---	---	1/5
2	110	115	90	1.10	1.09	1.13	1/5
3	115	120	95	1.05	1.04	1.06	1/5
4	120	125	105	1.04	1.04	1.11	1/5
5	125	130	130	1.04	1.04	1.24	1/5

Utility

Then:

$\overline{R}_A = 1.06; \overline{R}_B = 1.05; \overline{R}_C = 1.14$

$\sigma_A = 0.025; \sigma_B = 0.022; \sigma_C = 0.052$

$\overline{W}_A = 114; \overline{W}_B = 119; \overline{W}_C = 100$

If risk less interest (in terms of total return) is 0.5, then using mean-variance analysis we rank the assets as

$B\left\{\left[\left(\overline{R}_B - R_f\right)/\sigma_B\right] = 25.0\right\} > A\left\{\left[\left(\overline{R}_A - R_f\right)/\sigma_A\right] = 22.4\right\} > C\left\{\left[\left(\overline{R}_C - R_f\right)/\sigma_C\right] = 12.3\right\}$

Using quadratic utility function $U(W) = W - b*W^2$, with b = -0.002 we rank the assets as

B [U(B) = 90.68] > A [U(A) = 88.01] > C [U(C) = 80.00]

Utility

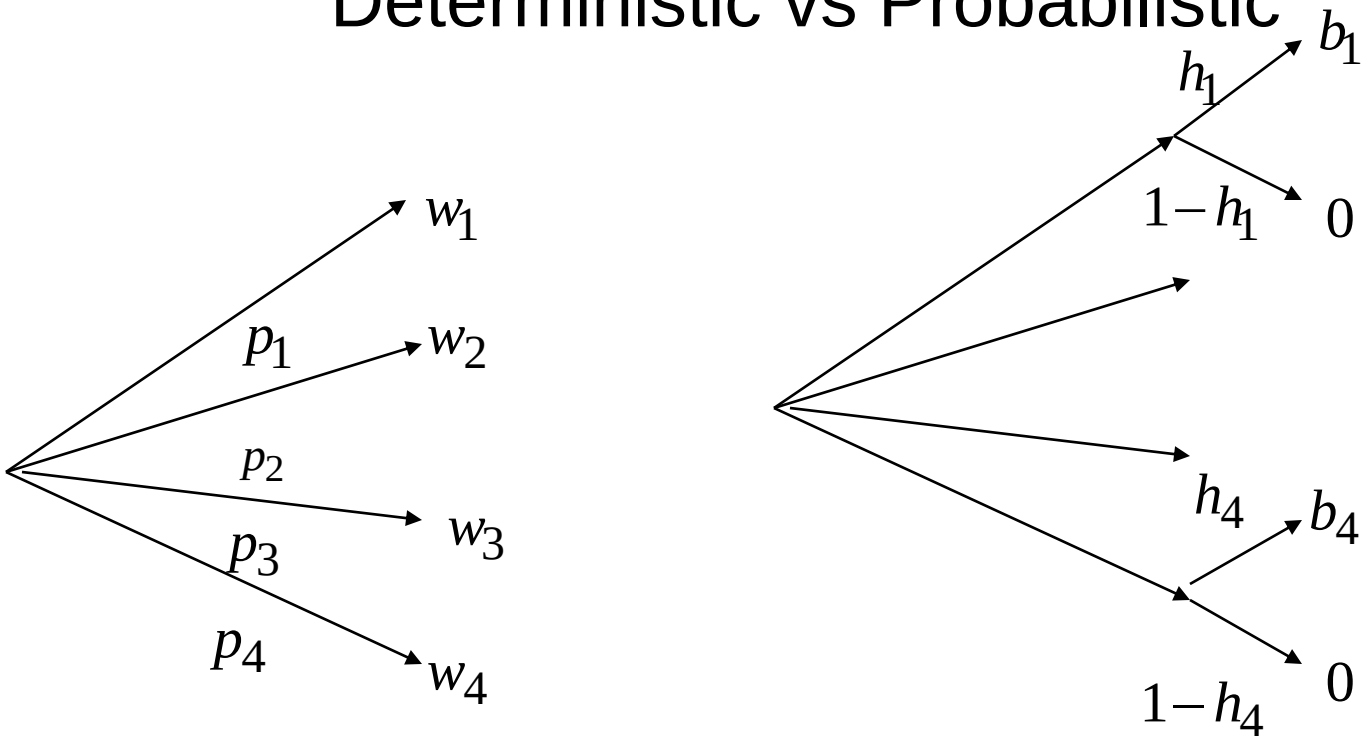
Consider the following example with two different sets of outcomes. The utility function is $U[W] = W^2 + W$

Outcome Scenario 1	Outcome Scenario 2	W	U[W]	P(W)
15	20	1.5	3.75	(15+20)/212
20	12	2.0	6.00	(20+12)/212
25	25	2.5	8.75	(25+25)/212
10	17	3.0	12.00	(10+17)212
5	8	3.5	15.75	(5+8)/212
25	30	4.0	20.00	(25+30)/212

Accordingly we have to calculate the expected utility value

Utility

Deterministic vs Probabilistic



Introduction

We must make a subtle distinction between investment and savings. While the former signifies a portion of wealth which is held for its (wealth) increase, while the latter is for consumption purposes only.

For investments we can have both real as well as financial investments, but for our study we will consider only financial investments.

Investment Process

Each group has to collect the data of (i) five scripts pertaining to their closing prices in NSE; (ii) S&P CNX Nifty; (iii) interest rate/yield for 91 days GOI T-Bill; (iv) INR vs USD rate; (v) INR vs UKP; (vi) Brent crude oil price (per barrel) in USD. All the data must be collected from 01-01-2005 to 31-07-2007.

Note: None of the groups should have any common scripts between themselves.

Definition

For borrowing money we always need some sort of collateral, against which we can get the money. The collaterals can be a house, jewelry, shares, a car, etc.

Security

In reality a security is just a piece of paper which represents the investor′s right over the property of the company. The investor may or may not exercise those rights depending on what he/she thinks about the future prospect of the company and how it will pay him/her benefit for holding this piece of paper (security). Remember for legally holding this piece of paper (security) the investor has to make payments for purchasing it.

Definition

The question which automatically comes to our mind, it how does one evaluate this investment of money (for purchasing this security).

For this we use the concept of (i) rate of return and (ii) total return

Rate of return (r) is calculated as:

$$r = (I_{t=T} - I_{t=0})/I_{t=o}$$

Where:

r = rate of return

$I_{t=T}$ = end of period (i.e., at time ′t=T′) wealth

$I_{t=0}$ = beginning of period (i.e., at time ′t=0′) wealth